N-body algorithms in GANDALF

David Hubber

USM, LMU, München
Excellence Cluster Universe,
Garching bei München, Germany

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Collisional vs. Collisionless N-body dynamics

- N-body algorithms are usually divided up into two main classes:
  - **Collisional**: N-body particles are central point masses which can have strong 2-body interactions (e.g. stellar encounters)
    - NBODY6, Starlab/kira
  - **Collisionless**: N-body particles have a smoothed potential so only feel long-range potential forces (e.g. cold-dark matter fluid)
    - GADGET 2/3, GASOLINE

- Both ‘versions’ of N-body simulations can be realised in GANDALF

- However, the collisional N-body dynamics is **only realised designed for relatively small N-body systems** and not for large-N systems (e.g. the million body problem)
Simple collisionless N-body integrators

**Collisionless** N-body integrators in GANDALF use the same algorithms as the SPH particles, i.e.

- Leapfrog kick-drift-kick (i.e. lfkdk)

\[
\mathbf{r}_i(t + \Delta t) = \mathbf{r}_i(t) + \mathbf{v}_i(t) \Delta t + \frac{1}{2} \mathbf{a}_i(t) \Delta t^2
\]

\[
\mathbf{v}_i(t + \Delta t) = \mathbf{v}_i(t) + \frac{1}{2} (\mathbf{a}_i(t) + \mathbf{a}_i(t + \Delta t)) \Delta t
\]

- Leapfrog drift-kick-drift (i.e. lfdkd)

- These integrators are **symplectic**, i.e. have very good conservation properties, particularly angular momentum
Simple collisionless N-body integrators

- **Simplest way** to simulate collisionless N-body is to use SPH particles with self gravity but hydro_forces switched off:

  \[
  \text{hydro}\_\text{forces} = 0 \\
  \text{self}\_\text{gravity} = 1
  \]

- Developing multi-species in GANDALF in order to have cdm particles, i.e. self-gravitiating but no hydro forces, as well as hydro particles

```c
template <int ndim>
struct Particle
{
  bool active;  ///< Flag if active (i.e. recompute step)
  bool potmin; ///< Is particle at a potential minima?
  int iorig;   ///< Original particle i.d.
  int itype;   ///< SPH particle type
  etc..
};
```

```c
part.itype = gas;
part.itype = cdm;
```
Simple collisional N-body integrators

- **Collisional** N-body integrators are more demanding because
  - Stars may have rather violent 2-body (or 3-body) interactions
  - Requires much higher accuracy with the integrations

- Simplest integrators are the same as the collisionless code
  - Leapfrog kick-drift-kick (i.e. lfkdk)
  - Leapfrog drift-kick-drift (i.e. lfdkd)
More sophisticated N-body integrators

• For more accuracy, we can use:
  • 4th, 6th and 8th-order Hermite scheme (Makino & Aarseth 1992)
  • KS-regularisation

• Hermite schemes compute both the force AND the force derivative

\[
a_s = -G \sum_{t=1}^{N} m_t \phi'(r_{st}, \bar{h}_{st}) \hat{r}_{st} - G \sum_{i=1}^{N} m_i \phi'(r_{si}, \bar{h}_{si}) \hat{r}_{si}
\]

\[
\dot{a}_s = -G \sum_{t=1}^{N} \frac{m_t \phi'(r_{st}, \bar{h}_{st})}{|r_{st}|} v_{st} + 3G \sum_{t=1}^{N} \frac{m_t (r_{st} \cdot v_{st}) \phi'(r_{st}, \bar{h}_{st})}{|r_{st}|^3} r_{st}
\]

\[
-4\pi G \sum_{t=1}^{N} \frac{m_t (r_{st} \cdot v_{st}) W(r_{st}, \bar{h}_{st})}{|r_{st}|^2} r_{st}.
\]
A simple example: A plummer sphere ($N = 100$)
Energy errors in N-body codes

Galactic Dynamics (Binney & Tremaine 2008)
What about ‘Regularisation’?

- **KS-Regularisation** is a powerful technique used in some N-body codes to:
  - (i) allow very accurate integration of very close 2-body encounters
  - (ii) therefore eliminate the need for softening/smoothing of grav. forces

- Some reasons not to use it
  - **Extremely** complicated
  - Hard to combine other physics (e.g. gas forces)
  - There are alternatives these days, not quite as accurate but much easier to implement

- Will I get hunted down by Sverre Aarseth if I don’t use it??
  - Hopefully not
Sub-systems

• If binary or higher-order multiple systems form, then the simulation may progress slower and slower

• **Spends a lot of CPU effort integrating the binary system** with short timesteps as the **rest of the simulation proceeds very slowly**

• Most of the time, **the binary motion can be isolated and simulated as a separate system** (with or without external perturbations)

• If a binary is identified (as in the previous slide), then
  • Binary motion is integrated separately
  • Rest of simulation interacts with centre-of-mass of binary
Hydrodynamics + N-body

- GANDALF employs a hybrid scheme for modelling the evolution of a gaseous stellar cluster
  - Gas is modelled with SPH particles using 2nd order Leapfrog scheme
  - N-body particles are modelled with 4th-order Hermite scheme
- Derived **coupling terms that maintains energy conservation**
Possible challenges to hybrid scheme

- **N-body codes usually require high accuracy** (e.g. total energy conserved to less than 0.001% accuracy), but hydro-codes usually operate with much higher error tolerances.

  A simple workaround has been proposed by [200]. For each and every simulation the conservation of energy, momentum and angular momentum should be monitored. Reducing the time step size and increasing the force accuracy, say, if a tree is used for gravity, should improve the conservation properties. A correct code should ensure conservation to better than 1% over several thousand time steps.

  Rosswog (2009)

- However, modern SPH schemes derived via Lagrangian mechanics can, in principle, conserve momentum, angular momentum and energy to rounding error given a robust integration scheme.
Errors in SPH/N-body codes

- Integration (truncation) error
  - SPH - 2nd-order Leapfrog
  - N-body - 4th-order Hermite
- Block timesteps
- Gravity tree errors

![Graph showing fractional energy error vs. timestep](image)
Gaseous Plummer spheres

- A Plummer sphere can be combined with a $n=5$ polytrope to produce a stable ‘gaseous cluster’.
Modelling star formation: Sink particles

- Modelling how low-density gas collapses into stars is a very expensive process
- Can perhaps investigate a single star in detail
- **Almost impossible with current capabilities to model a cluster of fully formed stars**

- Bate, Bonnel & Price (1995) introduced **dynamical sink particles**, to mimic the formation of a star and to capture the effects of any subsequent accretion
- Sinks are created like little black holes / vacuum cleaners that sweep up any gas that enters it
- Allows simulations to run fast enough to follow large-scale cluster formation
Sink particles: formation criteria

• The choice of formation criteria is crucial for obtaining converged simulations

• We use the following criteria

  • Exceeds a density threshold
    \[ \rho_i > \rho_{\text{SINK}} \]

  • Gravitational potential minima
    \[ \phi_i < \text{MIN} \{ \phi_j \} \]

  • Doesn’t overlap with existing sink
    \[ |\mathbf{r}_i - \mathbf{r}_{s'}| > X_{\text{SINK}} h_i + R_{s'} \]

• There’s an additional criterion which should be implemented soon

  • Hills sphere criteria
    \[ \rho_i > \rho_{\text{HILL}} \equiv \frac{3 X_{\text{HILL}} (-\mathbf{\Delta r}_{is'} \cdot \mathbf{\Delta a}_{is'})}{4 \pi G |\mathbf{\Delta r}_{is'}|^2} \]
Sink particles: formation criteria

Density criterion

Density & potential minimum criteria

Low sink density

High sink density
Modelling accretion

- Accretion is modelled by removing particles from the simulation that
  - Enter the sink accretion radius
  - Are gravitationally bound to the sink
- Generally leads to an empty ‘exclusion’ zone inside the sink that is devoid of any SPH particles
Artificial boundary forces

- Particles just outside the accretion radius see no neighbours inside the sink
- Discontinuous sampling of density field
- All SPH properties are incorrect, in particular the hydro forces
- Leads to artificial outward pressure gradient, and therefore artificial inward hydro force
- BBP95 originally suggested using some correction terms to account for missing neighbours
- Does not work so well and is not used any more (as far as I know)
Spherical and disc accretion

- For sub-grid accretion model, we consider two limiting cases

**Spherical accretion**

\[
\dot{M}(r) = -4\pi r^2 \rho(r) v_{\text{RAD}}(r).
\]

\[
\langle t_{\text{RAD}} \rangle_s = \frac{\sum_j \{m_j\} \mathcal{W}}{4\pi \sum_j \{\Delta r_{js} \mid \Delta r_{js} \cdot \Delta v_{js} m_j W(|\Delta r_{js}|, H_s)\}}
\]

**Disc accretion**

\[
t_{\text{SS}} = \frac{(G M_* R_d)^{1/2}}{\alpha_{\text{SS}} a^2}
\]

\[
\langle t_{\text{DISC}} \rangle = \frac{(G M_s)^{1/2}}{\alpha_{\text{SS}} \mathcal{W}} \sum_j \left\{ \frac{\mid\Delta r_{js}\mid^{1/2} m_j W(|\Delta r_{js}|, H_s)}{\rho_j a_j^2} \right\}
\]
Computing the accretion timescale

• We compute the ratio of rotational energy to gravitational energy of particles inside the sink as an indicator of which limiting case is applicable

\[ f = \MIN \left\{ \frac{2E_{\text{rot}}}{|E_{\text{grav}}|}, 1 \right\} \]

• If \( f = 1 \), particles are in **rotational equilibrium**:

\[ t_{\text{ACC}} \rightarrow \langle t_{\text{DISC}} \rangle_s \]

• If \( f = 0 \), particle motion is **purely radial**:

\[ t_{\text{ACC}} \rightarrow \langle t_{\text{RAD}} \rangle_s \]

• To deal with intermediate cases that also give the correct limiting behaviour, we use

\[ t_{\text{ACC}} = \langle t_{\text{RAD}} \rangle_s^{(1-f)} \langle t_{\text{DISC}} \rangle_s^f \]

• The mass accreted in the current timestep is then

\[ \delta M_{\text{ACC}} = M_{\text{INT}} \left[ 1 - \exp \left( -\frac{\delta t_s}{t_{\text{ACC}}} \right) \right] \]
Bondi accretion (Spherical accretion)

• In Bondi accretion, the sonic point defines the radius where the inflow velocity is equal to the local sound speed.

• For large radii, the inflow is subsonic (both hydro and gravity forces important).

• For small radii, the inflow is supersonic (only gravity important).

• Old sinks are correct for small radii since the lack of hydro forces is unimportant. For large radii, the lack of hydro forces leads to incorrect accretion rates.

• New sinks give correct accretion rates for all sink radii.

• Note: For monatomic gases, the sonic radius is zero. Therefore, for old sinks the accretion rate is always wrong.
Boss-Bodenheimer test: Convergence of sink properties

- For old sinks, the total mass contained in a sink varies greatly depending on the formation density, and hence the sink radius.
- For new sinks, although the results vary with resolution (external hydrodynamics), they are essentially independent of sink density/radius.
Boss-Bodenheimer test: Convergence of sink properties

- For larger sinks (same formation density), old sinks have even larger masses, but new sinks are still converged at the same masses
Future development

- GANDALF will allow both collisional and collisionless N-body simulations (but far more optimised for collisionless)
- Collisional N-body will be optimised in the future, particularly with the sub-systems and binary integrators
- Sink particles currently only implemented in SPH schemes
  - Will be added to Meshless scheme soon